

## Hintikka's Take on the Axiom of Choice and the Constructivist Challenge<sup>†</sup>

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### Resumen

En el presente trabajo confrontamos el análisis del axioma de elección de Martin - Löf con la posición de J. Hintikka respecto de este axioma. Hintikka afirma que su Semántica Teorética de Juegos (STJ) para una Lógica de la Independencia Amigable (Lógica IA), justifica el axioma de elección de Zermelo en un sentido de primer orden perfectamente aceptable para los constructivistas. De hecho, los resultados de Martin - Löf conducen a las siguientes consideraciones:

1) La versión preferida de Hintikka del axioma de elección, es ciertamente aceptable para los constructivistas y su significado no implica una lógica de orden superior.

2) Sin embargo, la versión aceptable para los constructivistas se basa en una consideración intensional sobre las funciones. La extensionalidad es el corazón de la comprensión clásica del axioma de Zermelo y esta es la razón real tras el rechazo constructivista de éste.

3) En general, las características de dependencia e independencia que motivan la Lógica IA, pueden formularse en el marco de la Teoría de Tipos Constructiva ( TTC ) sin tener que pagar el precio de un sistema que no es ni axiomatizable ni tiene una teoría subyacente de la inferencia – la lógica trata sobre la inferencia después de todo.

Concluimos señalando que los recientes desarrollos en lógica dialógica muestran que el enfoque TTC hacia el significado, en general, y hacia el axioma de elección, en particular, es connatural al enfoque de la teorética de juegos, donde las características metalógicas (standard) se despliegan explícitamente a nivel del lenguaje - objeto. Por tanto, de algún modo, esto justifica, aunque de una manera bastante diferente, la exhortación de Hintikka por la fecundidad de la Semántica Teorética de Juegos en el contexto de los fundamentos de las matemáticas.

**PALABRAS CLAVE:** Axioma de elección, Lógica de la Independencia Amigable, Semántica Teorética de Juegos, Teoría de Tipos Constructiva.

### Abstract

In the present paper we confront Martin- Löf's analysis of the axiom of choice with J. Hintikka's standing on this axiom. Hintikka claims that his game theoretical semantics

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(GTS) for Independence Friendly Logic (IF logic) justifies Zermelo's axiom of choice in a first-order way perfectly acceptable for the constructivists. In fact, Martin- Löf's results lead to the following considerations:

- 1) Hintikka preferred version of the axiom of choice is indeed acceptable for the constructivists and its meaning does not involve higher order logic
- 2) However, the version acceptable for constructivists is based on an intensional take on functions. Extensionality is the heart of the classical understanding of Zermelo's axiom and this is the real reason behind the constructivist rejection of it.
- 3) More generally, dependence and independence features that motivate IF-Logic, can be formulated within the frame of constructive type theory (CTT) without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference – logic is about inference after all.

We conclude pointing out that recent developments in dialogical logic show that the CTT approach to meaning in general and to the axiom of choice in particular is very natural to game theoretical approaches where (standard) metalogical features are explicitly displayed at the object language-level. Thus, in some way, this vindicates, albeit in quite of a different manner, Hintikka's plea for the fruitfulness of game-theoretical semantics in the context of the foundations of mathematics.

**KEY WORDS:** axiom of choice, independence friendly logic, game theoretical semantics, constructive type theory.

## 1 Introduction

In the present paper we confront Martin- Löf's analysis of the axiom of choice with Hintikka's standing on this axiom. Hintikka claims that his game theoretical semantics (GTS) for Independence Friendly Logic justifies Zermelo's axiom of choice in a first-order way perfectly acceptable for the constructivists. In fact, Martin- Löf's results lead to the following considerations:

- 1) Hintikka preferred version of the axiom of choice is indeed acceptable for the constructivists and its meaning does not involve higher order logic.
- 2) However, the version acceptable for constructivists is based on an intensional take on functions. Extensionality is the heart of the classical understanding of Zermelo's axiom and this is the real reason behind the constructivist rejection of it.
- 3) More generally, dependence and independence features that motivate IF-Logic, can be formulated within the frame of constructive type theory (CTT) without paying the price of a system that is neither axiomatizable nor has an underlying theory of inference – logic is about inference after all.

We conclude pointing out that recent developments in dialogical logic show that the CTT approach to meaning in general and to the axiom of choice in particular is very natural to game theoretical approaches where (standard) metalogical features are explicitly displayed at the object language-level. Thus, in some way, this vindicates, albeit in quite of a different manner, Hintikka's plea for the fruitfulness of game-theoretical semantics in the context of the foundations of mathematics. In fact, from the

dialogical point of view, those actions that constitute the meaning of logical constants, such as choices, are a crucial element of its full-fledged (local) semantics.

## 2 Independence Friendly Logic and its GTS Interpretation

Jaakko Hintikka (1996a) proposed a new approach to the foundations of mathematics making use of *independent friendly logic* (IF logic).<sup>1</sup> This logic is proposed to provide an alternative to higher order logics or set theory that are most often used for foundational purposes. In the context of IF, more quantifier dependencies and independencies can be expressed than in first-order logic. Its quantifiers range over individuals only; semantically IF first-order logic, however, has the same expressive power as existential second-order logic. IF logic lacks certain metaproperties that first-order logic has (axiomatizability, Tarski-type semantics, and a proof-theory). On the other hand, IF logic admits a self-applied truth-predicate – a property that first-order logic notoriously does not enjoy. The latter, leads Hintikka to plea for both an IF setting and a game theoretical semantics as a new approach for the foundations of mathematics. He rejects set theory because of its ontological commitment to higher-order entities and standard first-order logic because its lack of expressivity in relation to some kind of independence and dependence relations between quantifiers that are decisive for the understanding of mathematics. According to Hintikka, what we need is theory able to represent these kinds of dependences and independence within a setting where the quantifiers bound first order entities. Moreover, his point is that a game theoretical interpretation of IF accomplishes this double task.

### 2.1 IF-Logic

*Independence friendly first-order logic* (a.k.a. *IF first-order logic*, *IF logic*) was introduced by Hintikka and Sandu in their article 'Informational Independence as a Semantical Phenomenon' (1989); other early sources are Hintikka's booklet *Defining Truth, the Whole Truth, and Nothing but the Truth* (1991) and Sandu's Ph.D. thesis (1991). IF first-order logic is an extension of first-order logic, involving a specific syntactic device '/' (slash, independence indicator), which has at the object language level the same effect as the meta-level modifier 'but does not depend on'. A classic example is Henkin's branching quantifiers

$$\begin{array}{l} \forall z \exists u \\ \\ \forall x \exists y \end{array} \left. \vphantom{\begin{array}{l} \forall z \exists u \\ \\ \forall x \exists y \end{array}} \right\} S(x,y,z,u)$$

This can not be expressed with one, linearly disposed expression of classical first-order. However this can be done in IF in the following way:

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<sup>1</sup> For a thorough overview see Tulenheimo (2009).

$$\forall x \forall z (\exists y / \forall z) (\exists u / \forall x) S(x, y, z, u)$$

where the slashes indicates that  $\exists y$  ( $\exists u$ ) is independent of  $\forall z$  ( $\forall x$ )

In some cases, the slash doesn't contribute to anything that couldn't be expressed without it, but in others, it allows to express structural features that we would not otherwise be able to express in standard first-order logic. Walkoe (1970) showed that the expressive power of formulas with branching quantifiers is precisely that of existential second order logic. Independently, Walkoe (1970) and Enderton (1970) also showed that every existential second order sentence  $\Sigma_1^1$  is equivalent to second order truth or falsity condition of an IF sentence <sup>2</sup>. Thus, IF logic captures exactly the expressive power of Henkin's branching quantifiers.

Beside its expressivity power IF logic has been proven to enjoy the following metalogical properties: compactness, separation theorem, Beth's theorem, Löwenheim-Skolem theorem. (Sandu and Sevenster, 2011). As already mentioned, the price to pay for this way to enrich the expressivity of first-order logic is the lack of an underlying inference theory and of a complete axiomatization.

As mentioned above, one of the mayor claims of Hintikka is that IF is first-order and that this is stressed by a GTS approach where meaning is obtained through a semantic game between two players, *Verifier* and *Falsifier*, starting from the whole formula and descending to the atomic formulas, the truth of which is checked in the model – in other words, the attribution of meaning goes exactly the inverse way of standard Tarski-style semantics which proceeds "from inside out".<sup>3</sup>

## 2.2 Game Theoretical Semantics for IF

By 1960 appeared *Dialogical logic* developed by Paul Lorenzen and Kuno Lorenz, as a solution to some of the problems that arouse in Lorenzen's *Operative Logik* (1955).<sup>4</sup> Herewith, the epistemic turn initiated by the proof theoretic approaches was tackled with the notion of games that provided the dynamic features of the traditional dialectical reasoning. Inspired by Wittgenstein's *meaning as use* the basic idea of the dialogical approach to logic is that the meaning of the logical constants is given by the norms or rules for their use. The approach provides an alternative to both model theoretic and proof-theoretic semantics.

<sup>2</sup> Existential second order logic is a fragment of second order logic that consists of a formula in the form  $\exists x_1 \dots \exists x_n \Psi$ , where  $\exists x_1 \dots \exists x_n$  are second-order quantifiers and  $\Psi$  is a first-order formula.

<sup>3</sup> Cf. Feferman (2006) and Väänänen (2001) raised however the question whether IF logic is really the first-order logic? Tulenheimo (2009) provides some elements to defend Hintikka's view. Curiously, Sundholm (2013) shows that those dependences and independences that motivate Hintikka's introduction of IF can be formulated in CTT-first order logic.

<sup>4</sup> The main original papers are collected in Lorenzen and Lorenz (1978). Other papers have been collected more recently in Lorenz (2010a,b).

In 1968 Hintikka combined the model-theoretical and the game-based traditions by means of the development of what is now known as Game Theoretical Semantics (GTS) that, like in the dialogical framework, grounds the concepts of truth or validity on game-theoretic concepts, such as the existence of a winning strategy for a player, though differently to the dialogical framework it is build up on the notion of model.<sup>5</sup> Hintikka claims that his semantic games are exact codifications of language-games in Wittgenstein's sense, at least if one accepts that the activities associated with quantifiers are 'looking for' and 'finding'.

Semantic games for IF logic are classified in the literature as *extensive games with imperfect information*. The game is defined as follows:

*-Definition:* Let Eloise and Abelard be the players in a game. Eloise is initial verifier, trying to defend the sentence at stake and Abelard is initial falsifier, trying to deny it.

A semantic game  $G(\varphi)$  for the sentence  $\varphi$  begins with  $\varphi$ . The game is played in model  $M$  with a given language  $L$ . Through various stages of game, players will consider either the sentence  $\varphi$  or other the sentence  $\varphi_1$  obtained of it through the game. The game is played with well-defined rules.

$R\vee$  - disjunction - rule:  $G(\varphi_1 \vee \varphi_2)$  starts by the choice of a player in the role of verifier for  $\varphi_i$  ( $i = 1$  or  $2$ ). The game continues as  $G(\varphi_i)$ .

$R\wedge$  conjunction-rule:  $G(\varphi_1 \wedge \varphi_2)$  starts by the choice of a player in the role of falsifier for  $\varphi_i$  ( $i = 1$  or  $2$ ). The game continues as  $G(\varphi_i)$ .

$R\exists$  - rule for existential quantifier:  $G(\exists x Sx)$  starts by the choice of a player in the role of verifier of one member from the domain of  $M$  for  $x$ . If the name of the individual is  $a$ , the game is played as  $G(Sa)$ .

$R\forall$ -rule for universal quantifier:  $G(\forall x Sx)$  starts by the choice of a player in the role of falsifier of one member from the domain of  $M$  for  $x$ . If the name of the individual is  $a$ , the game is played as a  $G(Sa)$ .

$R\sim$ - rule for negation:  $G(\sim \varphi)$  is played the same as  $G(\varphi)$  except that players change their roles.

$R$ -atomic- rule for the atomic sentences: if  $A$  is an atomic sentence that is true, the verifier wins. If the sentence is false the falsifier wins.

Each application of the rules eliminates one logical constant, so that in a finite number of steps we come to the rule for atomic sentences. Truth of an atomic sentence is determined in the model  $M$  with respect to which  $G(\varphi)$  is played. That is allowed by the interpretation of all non-logical constant terms in the model. This interpretation is an integral part of the model  $M$  and it provides the meaning to the primitive symbols of a given interpreted first-order language.

Finally, here are the truth and falsity conditions for an IF formula.

*-Definition:* a) Formula  $\varphi$  is true in model  $M$  ( $M \models \varphi$ ) if and only if there is a winning strategy for Eloise in the game  $G(\varphi)$  played in  $M$ .

<sup>5</sup> Cf. Hintikka (1996a), Hintikka and Sandu (1997). See also Hintikka (2000) and in particular. Rahman and Tulenheimo (2006) studied the relation between dialogical logic and GTS.

b) Formula  $\phi$  is false in model  $\mathbb{M}$  ( $\mathbb{M} \models \neg \phi$ ) if and only if there is a winning strategy for Abelard in the game  $G(\phi)$  played in  $\mathbb{M}$ .

The players in a game are of course idealised agents who are always capable of finding a winning strategy if there is one, so the game doesn't depend of the intellectual capacity of the players.

What the slash sign is bringing into a play is the restriction on the information available to a player. When a player is to make a move for a subformula with a slash sign he must make it without knowing the values of the variables under the slash sign that have been already chosen by the opponent. This is what the game theoretical meaning of the slash amounts to. Classical first-order formulas can be interpreted as IF formulas where the set of variables under the slash sign is empty, in other words, when there is no restriction on the information available to a player.

Games for classical first order logic are *extensive games with perfect information*. Those games are always *determined*, as Gale- Stewart theorem holds for them (Gale, Stewart, 1953) In other words, one of the players always wins the game. This is not the case with games with imperfect information, so the law of excluded middle doesn't hold for IF logic. That can easily be seen on a simple example of a sentence  $\phi$ :  $\forall x(\exists y/\forall x) (x=y)$  and the model  $\mathbb{M} = \{0,1\}$ . For this sentence neither Abelard nor Eloise has a winning strategy, so it is undetermined. However, the failure of the excluded middle is not motivated by the same reasons as in intuitionism. It is rather described as the structural property of the model in the sense that undetermined truth-value indicates that not all types of functions exist in the model (Tulenheimo, 2009).

One of Hintikka's most cherished uses of the game theoretical approach for the foundations of mathematics is his analysis of the axiom of choice.

### 2.3 GTS and the Axiom of Choice

As mentioned above, in GTS the conditions of truth and falsity are determined by means of winning strategies. Strategies are expressed by a finite set of choice-functions or *Skolem functions*. The values of those functions indicate those individuals that the Verifier has to choose in his actions when the game is about an existential quantifier (or a disjunction) in order to win the game. Similar applies a for Falsifier's strategy when the game is about a universal quantifier (or a conjunction). Let us take the following example:

$$1) \forall x \exists y C(x,y)$$

This sentence is true if there is a winning strategy for Eloise. The winning strategy tells her how to select a value for  $y$ , in function of the value of  $x$  ( $f(x) = y$ ). If we wish to express the existence of such a strategy, we write the following:

$$2) \exists f \forall x C(x,f(x))$$

If we link 1) and 2) with a conditional, we obtain the following formulation of the *axiom of choice*.

$$3) \forall x \exists y C(x, y) \rightarrow \exists f \forall x C(x, f(x))$$

From the point of view of GTS, the truth of 3) - that expresses one version of the axiom of choice, is in fact derived from the very definition of truth. In fact, as I will discuss further on, it is related to the truth of the universal.

Let me point out that the axiom of choice is essential for GTS, since without it Tarski- style semantic and GTS for classical first order logic would not be equivalent. The reason is that strategies in GTS are understood as *deterministic* strategies that impose to verifier and falsifiers their choices, leaving no real options. Hodges (2013) sees it as a weakness in Hintikka's approach and thinks that a more natural way to conceptualise GTS would be to use *non-deterministic* strategies so that no a priori presumption of the axiom of choice is needed. However, Hintikka insists that there is nothing troubling with that axiom of choice and that it actually constitutes our conception of truth.

*This paradigm problem concerns the status of the axiom of choice. This axiom was firmly rejected by Brouwer and it was mooted in the controversies between the French intuitionists and their opponents....The axiom of choice is true. The idea of "choosing" or "finding" suitable individuals is systematised in what is known as game-theoretical semantics. For mathematicians, this semantics is no novelty, however, but little more than a regimentation and generalisation of the way of thinking that underlies mathematicians' classical (or perhaps I should say Weierstrassian) epsilon-delta analyses of the basic concepts of calculus, such as continuity and differentiation...*

*...To return to the usual axiom of choice, it is thus seen to be unproblematically true. How can any intuitionist deny the axiom of choice...? What can possibly go wrong here? Moreover, evoking the concept of knowledge, either in the form of epistemic logic or informally, does not seem to help an intuitionistic critic of the axiom of choice at all, either... The discussion of the axiom of choice between intuitionists and classicists has conducted at cross-purposes. It can only be dissolved by making distinction between knowing that and knowing what that neither party has made explicit. (Hintikka, 2001: 8, 9, 13 )*

*Zermelo did not begin to axiomatize set theory unselfishly from the goodness of his theoretical heart. His main purpose was to justify his well-ordering theorem. In practice, this largely meant to justify the axiom of choice. [ ... ]. But that is not the full story. Worse still: Zermelo's specific enterprise was unnecessary, in that the so- called axiom of choice turns out to be in the bottom a plain first-order logical principle. (Hintikka, 2011: 81)*

In fact as we will see below the last sentence is true: the truth of the axiom of choice involves first-order logic. However, in contrast to what Hintikka argues against intuitionists, the axiom of choice is true in the context of intuitionistic first order logic enriched with axioms of ZFC- set theory formulated in the style of CTT.

### 3 Martin-Löf on the Axiom of Choice

It is well known that this axiom was first introduced by Zermelo in 1904 in order to prove Cantor's theorem that every set can be rendered to be well ordered. Zermelo gave two formulations of this axiom, one in 1904 and a second one in 1908. It is the second formulation that is relevant for our discussion, since it is related to both, Hintikka's and Martin-Löf's formalization:

*A set  $S$  that can be decomposed into a set of disjoint parts  $A, B, C, \dots$  each of the containing at least one element, possesses at least one subset  $S_1$  having exactly one element with each of the parts  $A, B, C, \dots$  considered. (Zermelo, 1908: 266)*

The Axiom attracted immediately much attention and both of its formulations were criticized by constructivists such as Baire, Borel, Lebesgue and Brower. The first objections were related to the non-predicative character of the axiom, where a certain choice function was supposed to exist without showing constructively that it does. However, the axiom found its way into the ZFC set theory and was finally accepted by majority of mathematicians because of its usefulness in different branches of mathematics.

Martin-Löf produced a proof of the axiom in a constructivist setting bringing together two seemingly incompatible perspectives on this axiom, namely:

- 1 Bishop's surprising observation from 1967: *A choice function exists in constructive mathematics, because a choice is implied by the very meaning of existence.*
- 2 The proof by Diaconescu (in 1975) and by Goodman and Myhle (in 1978) that the Axiom of Choice implies Excluded Middle.<sup>6</sup>

In his paper of 2006, Martin-Löf shows that there are indeed some versions of the axiom of choice that are perfectly acceptable for a constructivist, namely one where the choice function is defined *intensionally*. In order to see this the axiom must be formulated within the frame of a CTT-setting. Indeed such a setting allows comparing the extensional and the intensional formulation of the axiom. It is in fact the extensional version that implies Excluded Middle, whereas the intensional version is compatible with Bishop's remark:

*[...] this is not visible within an extensional framework, like Zermelo-Fraenkel set theory, where all functions are by definition extensional." (Martin-Löf, 2006: 349)*

In CTT the truth of the axiom actually follows rather naturally from the meaning of the quantifiers. Take the proposition  $(\forall x: A) P(x)$  where  $P(x)$  is of the type proposition provided  $x$  is an element of the set  $A$ . If the proposition is true; then there is a proof for

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<sup>6</sup> Taken from Martin-Löf, 2006: 346.



it. Such a proof is in fact a function that for every element  $x$  of  $A$  renders a proof of  $B(x)$ . This is how Bishop's remark should be understood: the truth of a universal amounts to the existence of a proof, and this proof is a function. Thus, the truth of a universal, amount in the constructivist account, to the existence of a function. From this the proof of the axiom of choice can be developed quite straightforwardly. If we recall that in the CTT-setting:

the existence of a function from  $A$  to  $B$  amounts to the existence of proof-object for the universal  $\forall x : A \rightarrow B$ , and that

the proof of the proposition  $Bx$ , existentially quantified over the set  $A$  amounts to a pair such that the first element of the pair is an element of  $A$  and the second element of the pair is a proof of  $Bx$ ;

a full-fledged formulation of the axiom of choice – where we make explicit the set over which the existential quantifiers are defined - follows:

$$(\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\forall x : A) C(x, f(x))$$

The proof of Martin-Löf (1980: 50-51) is the following

*The usual argument in intuitionistic mathematics, based on the intuitionistic interpretation of the logical constants, is roughly: to prove  $(\exists x)(\exists y)C(x,y)$  assume that we have a proof of the antecedent. This means we have a method which, applied to an arbitrary  $x$ , yields a proof of  $(\exists y)C(x,y)$ . Let  $f$  be the method which, to an arbitrarily given  $x$ , assigns the first component of this pair. Then  $C(x,f(x))$  holds for an arbitrary  $x$ , and hence, so does the consequent. The same idea can be put into symbols getting a formal proof in intuitionistic type theory. Let  $A$  : set,  $B(x)$ : set ( $x : A$ ),  $C(x,y)$ : set ( $x : A$ ,  $y : B(x)$ ), and assume  $z$ :  $(\forall x : A)(\exists y : B(x))C(x,y)$ . If  $x$  is an arbitrary element of  $A$ , i.e.  $x : A$ , then by  $\exists$ -elimination we obtain*

$$Ap(z,x) : (\exists y : B(x))C(x,y)$$

*We now apply left projection to obtain*

$$p(Ap(z,x)) : B(x)$$

*and right projection to obtain*

$$q(Ap(z,x)) : C(x,p(Ap(z,x))).$$

*By  $\exists$ -abstraction on  $x$  (or  $\exists$ -introduction), discharging  $x : A$ , we have*

$$(\exists x) p(Ap(z,x)) : (\exists x : A) B(x)$$

*and by  $=$ -equality*

$$Ap((x) p(Ap(z,x), x) = p(Ap(z,x))) : Bx.$$

By substitution [making use of  $C(x,y)$ : set  $(x: A, y: B(x))$ ,] we get

$$C(x, Ap((x) p(Ap(z,x), x) = p(Ap(z,x))))$$

[that is,  $C(x, Ap((x) p(Ap(z,x), x) = p(Ap(z,x))))$ : set ]

and hence by equality of sets

$$q(Ap(z,x)) : C(x, Ap((x) p(Ap(z,x), x)$$

where  $((x) p(Ap(z,x)))$  is independent of  $x$ . By abstraction on  $x$

$$((x) p(Ap(z,x))) : (\lambda x: A) C(x, Ap((x) p(Ap(z,x), x)$$

We now use the rule of pairing (that is  $\lambda$ -introduction) to get

$$(\lambda x: A) p(Ap(z,x)), (\lambda x: A) q(Ap(z,x)) : (\lambda f: (\lambda x: A) B(x)) (\lambda x: A) C(x, Ap(f,x))$$

(note that in the last step, the new variable  $f$  is introduced and substituted for  $((x) p(Ap(z,x)))$  in the right member). Finally by abstraction on  $z$ , we obtain

$$(\lambda z)((\lambda x: A) p(Ap(z,x))), ((\lambda x: A) q(Ap(z,x))) : (\lambda f: (\lambda x: A) B(x)) (\lambda x: A) C(x, Ap(f,x)).$$

Curiously this seems to be close to Hintikka's own formulation and even to his analysis that a winning strategy for a universal amounts to the existence of a (Skolem) function. It is curious since Martin-Löf's is developed within a constructivist setting. Moreover, Martin-Löf (2006: 347) shows what is wrong with the axiom from the constructivist point of view: it is its extensional formulation. That is:

$$(\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\text{Ext}(f) \ \& \ (\forall x : A) C(x, f(x)))$$

$$\text{Where } (\text{Ext}(f) = ((\forall i,j : A) (i =_A j \rightarrow f(i) = f(j)))$$

Thus, from the constructivist point of view; what is really wrong with the classical formulation of the axiom of choice is the assumption that from the truth that all of the  $A$  are  $B$  we can obtain a function that satisfies extensionality. In fact, as shown by Martin-Löf (2006: 349), the classical version holds, even constructively, if we assume that there is only one such choice function in the set at stake!

$$(\forall x : A) (\exists ! y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\text{Ext}(f) \ \& \ (\forall x : A) C(x, f(x)))$$

Let us retain that

- If we take  $(\forall x : A) (\exists y : Bx) C(x,y) \rightarrow (\exists f : (\forall x : A) Bx) (\forall x : A) C(x, f(x))$  to be the formalization of the axiom of choice, then that axiom is not only unproblematic for constructivists but it is also a theorem. But this formalization is a full-fledged formulation of the version Hintikka's adopts.<sup>7</sup> Certainly, the point is that the CTT-formulation stresses explicitly that the choice function at stake has been defined by means of intensional equality but Hintikka seems to assume extensionality. In fact it is the CTT-explicit language that allows a fine-grained distinction between the, on the surface, equivalent formulations. This is due to the expressive power of CTT that allows to express at the object language level properties that in other settings are left implicit in the metalanguage. This leads us to the second point
- According to the constructivist approach functions are identified as proof-objects for propositions and are given in *object-language*, as the objects of a certain type. Understood in that way, functions belong to the lowest-level of entities and there is no jumping to higher order. Once more, the truth of a first order-universal sentence, amounts to the existence of a function that is defined by means of the elements of the set over which the universal quantifies and the first-order expression  $Bx$ . The existence of such a function is the CTT-way to express at the object language level, that a given universal sentence is true.

Thus, Hintikka is right in defending that we need only first-order language, but this does not really support his attachment to the classical understanding of it. But what about his claim of the importance of GTS? This takes us to the next and final chapter.

#### 4 Conclusion: The Antirealist Rejoinder of Hintikka's Take on the Axiom of Choice

According to Hintikka, his preferred formulation of the axiom of choice, namely  $\forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x,f(x))$  (where it is left implicit that  $\forall x$  quantifies over, say the set  $A$ ,  $\exists y$  quantifies over, say the set  $B$ , and  $\exists f$ , over the set  $((\forall x : A) Bx)$  is perfectly acceptable for the constructivists. Let us recall, that the GTS reading of its truth amounts to the existence of winning strategy for Eloise in a game  $G(\forall x \exists y C(x,y))$ . The latter amounts to finding "witness individual"  $y$  dependent upon  $x$ , such that  $C(x,y)$  is true. In other words, the existence of a winning strategy for that game provides *a proof that the proposition  $S(x,y)$  is true in the model*. Hintikka claims that it is the GTS reading that makes the axiom of choice acceptable for the constructivists:

*Moreover, the rules of semantical games should likewise be acceptable to a constructivist. In order to verify an existential sentence  $\exists x S[x]$  I have to find an individual  $b$  such that I can (win in the game played with)  $S[b]$ . What could be a more constructivistic requirement than that? Likewise, in the verification game  $G(S_1 \vee S_2)$  connected with a disjunction  $(S_1 \vee S_2)$ , the verifier must choose  $S_1$  or  $S_2$  such that the game connected with it (i.e.,  $G(S_1)$  or  $G(S_2)$ ) can be won*

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<sup>7</sup> Indeed, Martin-Löf's formalization follows from making explicit in Hintikka's formulation  $\forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x,f(x))$  the range of its quantifiers, that is:  $\forall x$  quantifies over, say the set  $A$ ,  $\exists y$  quantifies over, say the set  $Bx$ , and  $\exists f$ , over the set  $(\forall x : A) Bx$ .

*by the verifier. Again, there does not seem to be anything here to alienate a constructivist.* (Hintikka, 1996: 212)

As mentioned above Hintikka is right, this is acceptable for the constructivists, but the reason is the underlying intensionality of the choice function. Hintikka, would like to have a logic that renders classical mathematics. However; this is only possible if we assume extensionality. Now to have both extensionality and the truth of the axiom of choice – without assuming unicity of the function – seems not to be possible. Or, if possible we need something like the IF-setting that is not completely axiomatisable. The price to pay is certainly high.

Hintikka's intention was to offer a realist foundation of mathematics on a first order level in a way that all classical mathematics can be comprised and that it can still be acceptable for a constructivist. Preceding analysis shows that it amounts to having a cake and eating it too. As pointed out by Göran Sundholm (2013) under constructivist reading IF logic is granted to be a first order logic, but in that case not all of the classical mathematics can be saved. Moreover if we comply with the constructive reading of IF logic it makes the whole endeavour somewhat superfluous. Then we can use some version of constructive logic without going into the whole trouble with one non- axiomatisable logic.

However, I think that there is valuable point in a game theoretical reading, that has been stressed by recent work on dialogical approaches to CTT by Shahid Rahman and Nicolas Clerbout (2013a, 2013b and 2014), who by the way worked out a dialogical proof of the constructive formulation of the axiom of choice. Indeed, if meaning is conceived as being constituted during interaction, then all of the actions involved in the constitution of the meaning of an expression should be rendered explicit. They should all be part of the object language. The roots of this perspective are based on Wittgenstein's *Un-Hintergebarkeit der Sprache*. Language-games are purported to accomplish the task of displaying this “internalist feature of meaning”. Indeed, one of the main insights of Lorenz' interpretation of the relation between the so-called *first* and *second* Wittgenstein is based on a thorough criticism of the metalogical approach to meaning (Lorenz 1970: 74-79). As pointed out by Lorenz, the heart of Wittgenstein's philosophy of language is the internal relation between language and world. The internal relation is what language games display while they constitute meaning: there is no way to ground a logical language outside language (recall the case of Neurath's sailor in his raft):

*Also propositions of the metalanguage require the understanding of propositions, [...] and thus can not in a sensible way have this same understanding as their proper object. The thesis that a property of a propositional sentence must always be internal, therefore amounts to articulating the insight that in propositions about a propositional sentence this same propositional sentence does not express anymore a meaningful proposition, since in this case it is not the propositional sentence that it is asserted but something about it.*<sup>8</sup>

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<sup>8</sup> Similar criticism has been raised by G. Sundholm (1997, 2001) who points out that the standard model-theoretic approaches to meaning turn semantics into a meta-mathematical formal object where syntax is linked

*Thus, if the original assertion (i.e., the proposition of the ground-level) should not be abrogated, then this same proposition should not be the object of a metaproposition, [...].*<sup>9</sup> (Lorenz, 1970: 75).

*While originally the semantics developed by the picture theory of language aimed at determining unambiguously the rules of "logical syntax" (i.e. the logical form of linguistic expressions) and thus to justify them [...] – now language use itself, without the mediation of theoretic constructions, merely via "language games", should be sufficient to introduce the talk about "meanings" in such a way that they supplement the syntactic rules for the use of ordinary language expressions (superficial grammar) with semantic rules that capture the understanding of these expressions (deep grammar).*<sup>10</sup> (Lorenz, 1970: 109).

If we recall Hintikka's (1996b) extension of van Heijenoort distinction of a *language as the universal medium* and *language as a calculus*, the point is that the dialogical approach shares some tenets of both conceptions. Indeed on one hand the dialogical approach shares with universalists the view that we cannot place ourselves outside our language, on the other it shares with the anti-universalists the view that we can develop a methodical reconstruction of a given complex linguistic practice out of the interaction of simple ones.

Unfortunately, this one of the wittgenstian perspectives that Hintikka explicitly rejects. Nevertheless, it is exactly this feature of the game theoretical approach to the axiom of choice that makes the reading acceptable. The game theoretical reading of the axiom of choice stresses one of the more salient characteristics of the CTT language: the judgement that a proposition is true can be expressed at the language level. The existence of a winning strategy is part of the first-order language indeed – as pursued by the introduction of a truth predicate in IF logic. Moreover, this explicit theory of meaning is

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to semantics by the assignation of truth values to uninterpreted strings of signs (formulae). Language does not any more *express content* but it is rather conceived as a system of signs that speaks *about* the world - provided a suitable metalogical link between signs and world has been fixed.

<sup>9</sup> The quote is a translation by S. Rahman of the following original text:

Auch Metaaussagen, so können wir zusammenfassen sind auf das Verständnis von Aussagen, [...] angewiesen, und können dieses Verständnis nicht sinnvoll zu ihrem Gegenstand machen. Die These, dass eine Eigenschaft eines Aussagesatzes stets intern sein muss, besagt daher nichts anderes, als die Artikulation der Einsicht, dass in Aussagen über einen Aussagesatz selbst nicht mehr der Ausdruck einer sinnvollen Aussage ist, nicht er wird behauptet, sondern etwas über ihn.

Wenn also die originale Behauptung, die Aussage der Grundstufe nicht ausser Kraft gesetzt werden soll, darf sie nicht zum Gegenstand einer Metaaussage gemacht werden, [...]. (Lorenz, 1970: 75).

<sup>10</sup> The quote is a translation by S. Rahman of the following original text:

Diente ursprünglich die mit der Abbildtheorie entworfene Semantik dazu, die Regeln der 'logischen Syntax', also die logische Form sprachlicher Ausdrücke, eindeutig zu bestimmen und damit zu rechtfertigen [...]-, so soll jetzt der Sprachgebrauch selbst, ohne Vermittlung theoretischer Konstruktionen, allein auf dem Wege über die 'Sprachspiele', zur Einführung der Rede von 'Bedeutungen' hinreichen und die syntaktischen Regeln zur Verwendung gebrauchsprachlicher Ausdrücke (Oberflächengrammatik) mit semantischen, das Verständnis dieser Ausdrücke darstellenden Regeln (Tiefengrammatik), ergänzen. (Lorenz, 1970: 109).

neutral in relation to classical or constructive logic. However, the proof of the axiom of choice is constructive and its game theoretical interpretation is antirealist after all!

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